

Área bajo la Curva e Integral Definida

4.5.4 Definición del área de una región plana

Sea f una función continua en $[a, b]$ y $f(x) \geq 0$ para toda x en $[a, b]$. Sea R la región limitada por la curva $y = f(x)$, el eje x y las rectas $x = a$ y $x = b$. Entonces la medida A del área de la región R está dada por

$$A = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(w_i) \Delta_i x$$

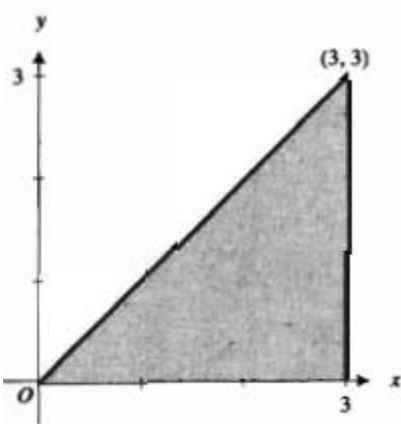
$$= \int_a^b f(x) dx$$

► **EJEMPLO 1** Calcule el valor de cada una de las siguientes integrales definidas interpretándolas como la medida del área de una región plana: (a) $\int_0^3 x dx$; (b) $\int_{-3}^3 \sqrt{9 - x^2} dx$; (c) $\int_{-2}^2 (2 - |x|) dx$.

Solución

(a) Con $f(x) = x$, la figura 5 muestra la región triangular limitada superiormente por la gráfica de f , inferiormente por el eje x , y por la derecha por la recta $x = 3$. De la fórmula para el área de un triángulo, el número de unidades cuadradas del área es $\frac{1}{2}(3)(3) = \frac{9}{2}$. Por tanto,

$$\int_0^3 x dx = \frac{9}{2}$$

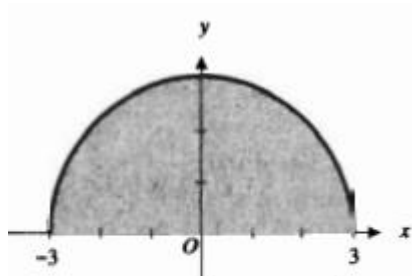


$$f(x) = x$$

FIGURA 5

(b) El integrando de la integral definida dada es $\sqrt{9 - x^2}$, el cual se denota por $g(x)$. La gráfica de g es una semicircunferencia con centro en el origen y radio 3. La figura 6 muestra la región limitada por arriba por esta semicircunferencia y por debajo por el eje x , en el intervalo $[-3, 3]$. El área de esta región es la mitad del área de la región encerrada por la circunferencia completa. Como el área de la región circular está dada por πr^2 , se tiene

$$\begin{aligned} \int_{-3}^3 \sqrt{9 - x^2} dx &= \frac{1}{2} \pi (3)^2 \\ &= \frac{9}{2} \pi \end{aligned}$$

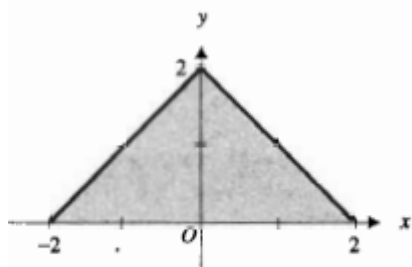


$$g(x) = \sqrt{9 - x^2}$$

FIGURA 6

(c) Con $h(x) = 2 - |x|$, se dibuja la gráfica de h en el intervalo $[-2, 2]$ y se obtiene la figura 7. Observe que la región limitada por la gráfica de h y el eje x es un triángulo de base 4 y altura 2. Como el número de unidades cuadradas del área de esta región triangular es $\frac{1}{2}(4)(2) = 4$, se tiene

$$\int_{-2}^2 (2 - |x|) dx = 4$$



$$h(x) = 2 - |x|$$

FIGURA 7

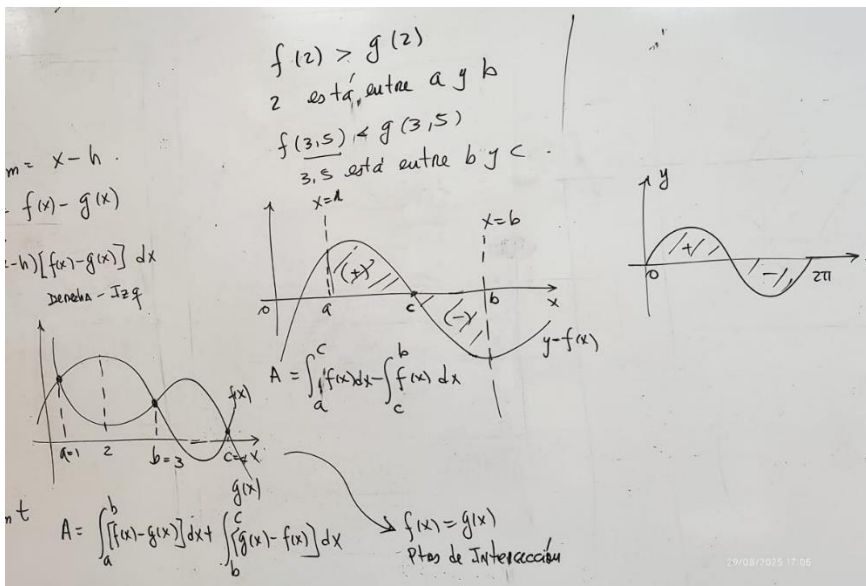
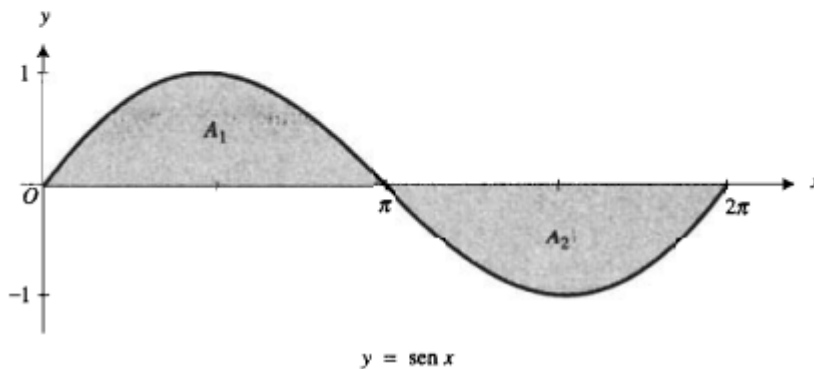
► **EJEMPLO 2** Obtenga una aproximación de $\int_0^{2\pi} \text{sen } x \, dx$ calculando $\text{NINT}(\text{sen } x, 0, 2\pi)$. Interprete la respuesta en términos de área.

Solución En la graficadora, $\text{NINT}(\text{sen } x, 0, 2\pi) = 0$. Así

$$\int_0^{2\pi} \text{sen } x \, dx = 0$$

La gráfica de la función seno de 0 a 2π se muestra en la figura 8. Sean A_1 unidades cuadradas y A_2 unidades cuadradas las áreas de las regiones limitadas por la curva senoidal y el eje x en los intervalos $[0, \pi]$ y $[\pi, 2\pi]$, respectivamente. Entonces, como A_1 y A_2 son iguales, se tiene

$$\int_0^{2\pi} \text{sen } x \, dx = A_1 - A_2 = 0$$



Problemas

13. $\int_0^2 \sqrt{4 - x^2} dx$

14. $\int_{-4}^4 \sqrt{16 - x^2} dx$

15. $\int_{-2}^4 x dx$

16. $\int_0^4 (3 - x) dx$

17. $\int_{-1}^2 (5 - 2x) dx$

18. $\int_{-1}^2 (2x + 5) dx$

19. $\int_{-2}^4 |x| dx$

20. $\int_{-3}^3 |1 - x| dx$

21. $\int_0^5 (|x + 3| - 5) dx$

22. $\int_{-1}^7 (|x - 2| - 3) dx$

23. $\int_0^8 (6 - |x - 2|) dx$

24. $\int_{-5}^0 (3 + |x + 4|) dx$

25. $\int_0^2 \sqrt{2x - x^2} dx$

26. $\int_{-1}^5 \sqrt{5 + 4x - x^2} dx$

27. $\int_{-\pi}^{\pi} \cos x dx$

28. $\int_{\pi/2}^{5\pi/2} \text{sen } x dx$

Solucionario

$$13. \int_0^2 \sqrt{4-x^2} dx = \text{area}(\text{first quadrant part of circle centered at } (0,0) \text{ of radius } 2) = \frac{1}{4}\pi(2)^2 = \pi$$

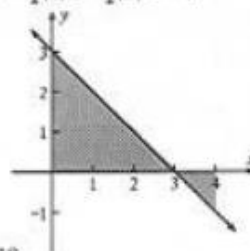
$$14. \int_{-4}^4 \sqrt{16-x^2} dx = \text{area}(\text{upper half of circle centered at } (0,0) \text{ of radius } 4) = \frac{1}{2}\pi(4)^2 = 8\pi$$

$$15. \int_{-2}^4 x dx = \text{area}(\text{triangle } (0,0), (4,0), (4,4)) - \text{area}(\text{triangle } (0,0), (-2,0), (-2,-2)) = \frac{1}{2}(4)4 - \frac{1}{2}(2)2 = 6$$

$$16. \int_0^4 (3-x) dx$$

> Let $f(x) = 3 - x$. Then $f(0) = 3$, $f(3) = 0$ and $f(4) = -1$. Hence the value of the integral is the measure of the area of the triangle $(0,0)$, $(0,3)$, $(3,0)$ above the x axis minus the measure of the area of the triangle $(3,0)$, $(4,0)$, $(4,-1)$ below the x axis. Therefore

$$\int_0^4 (3-x) dx = \frac{1}{2}(3)3 - \frac{1}{2}(1)1 = \frac{9}{2} - \frac{1}{2} = 4$$



$$17. \int_{-1}^2 (5-2x) dx = \text{area}(\text{trapezoid } (-1,0), (-1,7), (2,1), (2,0)) = \frac{1}{2}(7+1)[2 - (-1)] = 12$$

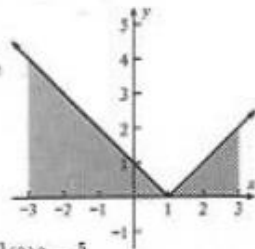
$$18. \int_{-1}^2 (2x+5) dx = \text{area}(\text{trapezoid } (-1,0), (-1,3), (2,9), (2,0)) = \frac{1}{2}(3+9)[2 - (-1)] = 18$$

$$19. \int_{-2}^4 |x| dx = \text{area}(\text{triangle } (0,0), (-2,0), (-2,-2)) + \text{area}(\text{triangle } (0,0), (4,0), (4,4)) = \frac{1}{2}(2)2 + \frac{1}{2}(4)4 = 10$$

$$20. \int_{-3}^3 |1-x| dx$$

> Let $f(x) = |1-x|$. Then $f(-3) = 4$, $f(1) = 0$ and $f(3) = 2$. Hence the value of the integral is the measure of the area of triangle $(1,0)$, $(-3,0)$, $(-3,4)$ plus the measure of the area of triangle $(1,0)$, $(3,0)$, $(3,2)$. Therefore

$$\int_{-3}^3 |1-x| dx = \frac{1}{2}(4)4 + \frac{1}{2}(2)2 = 10$$



$$21. \int_0^5 (|x+3|-5) dx = \int_0^5 [(x+3)-5] dx = \int_0^5 (x-2) dx$$

$$= \text{area}(\text{triangle } (2,0), (5,0), (5,3)) - \text{area}(\text{triangle } (2,0), (0,0), (0,-2)) = \frac{1}{2}(3)3 - \frac{1}{2}(2)2 = \frac{5}{2}$$

$$22. \text{ Let } f(x) = |x-2|-3. \text{ Then } f(-1) = 3-3=0, f(2) = 0-3=-3, f(5) = 3-3=0, f(7) = 5-3=2.$$

$$\int_{-1}^7 (|x-2|-3) dx = \text{area}(\text{triangle } (5,0), (7,0), (7,2)) - \text{area}(\text{triangle } (5,0), (2,-3), (-1,0))$$

$$= \frac{1}{2}(2)(2) - \frac{1}{2}(6)3 = -7$$

$$23. \text{ Let } f(x) = 6-|x-2|. \text{ Then } f(0) = 6-2=4, f(2) = 6-0=6, f(8) = 6-6=0.$$

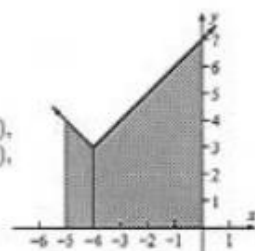
$$\int_0^8 (6-|x-2|) dx = \text{area}(\text{trapezoid } (0,0), (0,4), (2,6), (2,0)) + \text{area}(\text{triangle } (2,6), (2,0), (8,0))$$

$$= \frac{1}{2}(4+6)(2) + \frac{1}{2}(6)6 = 28$$

$$24. \int_{-5}^0 (3+|x+4|) dx$$

> Let $f(x) = 3+|x+4|$. $f(-5) = 3+1=4$, $f(-4) = 3+0=3$, $f(0) = 3+4=7$. Hence the value of the integral is the measure of the area of trapezoid $(-5,0)$, $(-5,4)$, $(-4,3)$, $(-4,0)$ plus the measure of the area of trapezoid $(-4,0)$, $(-4,3)$, $(0,7)$, $(0,0)$. Therefore

$$\int_{-5}^0 (3+|x+4|) dx = \frac{1}{2}(4+3)(1) + \frac{1}{2}(3+7)4 = \frac{47}{2}$$



$$25. \text{ Because } y^2 = 2x - x^2 = 1 - (x^2 - 2x + 1) = 1 - (x-1)^2; (x-1)^2 + y^2 = 1, \text{ then}$$

$$\int_0^2 \sqrt{2x-x^2} dx = \text{area}(\text{semicircle centered at } (1,0) \text{ of radius } 1) = \frac{1}{2}\pi(1)^2 = \frac{1}{2}\pi$$

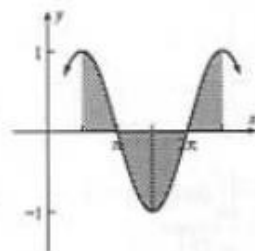
$$26. \text{ Because } y^2 = 5+4x-x^2 = 9 - (x^2 - 4x + 4) = 9 - (x-2)^2; (x-2)^2 + y^2 = 9,$$

$$\int_{-1}^5 \sqrt{5+4x-x^2} dx = \text{area}(\text{semicircle centered at } (2,0) \text{ radius } 3) = \frac{1}{2}\pi(3)^2 = \frac{9}{2}\pi$$

$$27. \int_{-\pi}^{\pi} \cos x dx = 0 \text{ because each region below the } x \text{ axis is congruent to one above.}$$

$$28. \int_{\pi/2}^{5\pi/2} \sin x dx$$

> Because each region above the x axis is congruent to one below, the integral is 0.



Área de la región entre curvas

► **EJEMPLO 4** Calcule el área de la región limitada por las curvas $y = x^2$ y $y = -x^2 + 4x$.

Solución Para determinar los puntos de intersección de las dos curvas se resuelven las ecuaciones simultáneamente y se obtienen los puntos $(0, 0)$ y $(2, 4)$. La figura 5 muestra la región.

Sean

$$f(x) = -x^2 + 4x \quad \text{y} \quad g(x) = x^2$$

Si A unidades cuadradas es el área de la región, entonces

$$\begin{aligned} A &= \int_0^2 [f(x) - g(x)] dx \\ &= \int_0^2 [(-x^2 + 4x) - x^2] dx \\ &= \int_0^2 (-2x^2 + 4x) dx \\ &= -\frac{2}{3}x^3 + 2x^2 \Big|_0^2 \\ &= -\frac{16}{3} + 8 - 0 \\ &= \frac{8}{3} \end{aligned}$$

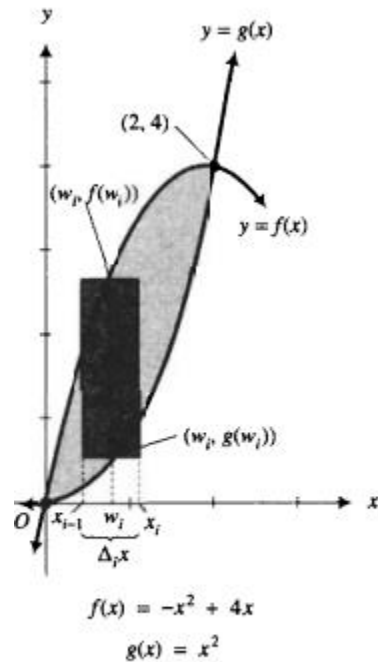
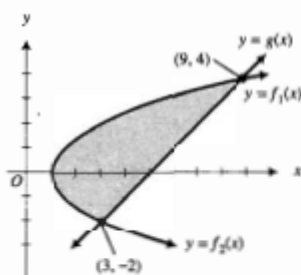


FIGURA 5

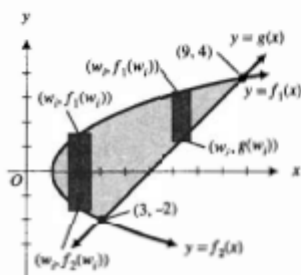


$$f_1(x) = \sqrt{2x-2}$$

$$f_2(x) = -\sqrt{2x-2}$$

$$g(x) = x - 5$$

FIGURA 6

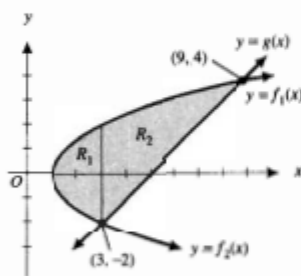


$$f_1(x) = \sqrt{2x-2}$$

$$f_2(x) = -\sqrt{2x-2}$$

$$g(x) = x - 5$$

FIGURA 7



$$f_1(x) = \sqrt{2x-2}$$

$$f_2(x) = -\sqrt{2x-2}$$

$$g(x) = x - 5$$

FIGURA 8

EJEMPLO 5 Calcule el área de la región limitada por la parábola $y^2 = 2x - 2$ y la recta $y = x - 5$.

Solución Las dos curvas se intersectan en los puntos $(3, -2)$ y $(9, 4)$. La región se muestra en la figura 6.

La ecuación $y^2 = 2x - 2$ es equivalente a las dos ecuaciones

$$y = \sqrt{2x-2} \quad \text{y} \quad y = -\sqrt{2x-2}$$

de modo que la primera ecuación proporciona la parte superior de la parábola mientras que la segunda ecuación da la parte inferior. Si

$$f_1(x) = \sqrt{2x-2} \quad \text{y} \quad f_2(x) = -\sqrt{2x-2}$$

la ecuación de la parte superior de la parábola es $y = f_1(x)$, y la ecuación de la parte inferior es $y = f_2(x)$. Si se considera que $g(x) = x - 5$, entonces la ecuación de la recta es $y = g(x)$.

En la figura 7 se aprecian dos elementos rectangulares verticales de área. Cada rectángulo tiene su base superior sobre la curva $y = f_1(x)$. Como la base inferior del primer rectángulo está sobre la curva $y = f_2(x)$, su altura es $[f_1(w_i) - f_2(w_i)]$ unidades. Debido a que la base inferior del segundo rectángulo está sobre la curva $y = g(x)$, su altura es $[f_1(w_i) - g(w_i)]$ unidades. Si se desea resolver este problema utilizando elementos rectangulares verticales de área, se debe dividir la región en dos regiones separadas, por ejemplo, R_1 y R_2 , donde R_1 es la región limitada por las curvas $y = f_1(x)$, $y = f_2(x)$ y la recta $x = 3$, y R_2 es la región limitada por las curvas $y = f_1(x)$ y $y = g(x)$ y la recta $x = 3$ (consulte la figura 8).

Si A_1 unidades cuadradas es el área de la región R_1 , entonces

$$A_1 = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [f_1(w_i) - f_2(w_i)] \Delta_i x$$

$$= \int_1^3 [f_1(x) - f_2(x)] dx$$

$$= \int_1^3 [\sqrt{2x-2} + \sqrt{2x-2}] dx$$

$$= 2 \int_1^3 \sqrt{2x-2} dx$$

$$= \frac{2}{3} (2x-2)^{3/2} \Big|_1^3$$

$$= \frac{16}{3}$$

Si A_2 unidades cuadradas es el área de la región R_2 ,

$$A_2 = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [f_1(w_i) - g(w_i)] \Delta_i x$$

$$= \int_3^9 [f_1(x) - g(x)] dx$$

$$= \int_3^9 [\sqrt{2x-2} - (x-5)] dx$$

$$= \frac{1}{3} (2x-2)^{3/2} - \frac{1}{2} x^2 + 5x \Big|_3^9$$

$$= \left[\frac{64}{3} - \frac{81}{2} + 45 \right] - \left[\frac{8}{3} - \frac{9}{2} + 15 \right]$$

$$= \frac{38}{3}$$

Entonces $A_1 + A_2 = \frac{16}{3} + \frac{38}{3}$.

► **EJEMPLO 6** Calcule el área de la región del ejemplo 5 considerando elementos rectangulares horizontales de área.

Solución La figura 9 muestra la región con un elemento rectangular horizontal de área.

Si las ecuaciones de la parábola y de la recta se resuelven para x se obtiene

$$x = \frac{1}{2}(y^2 + 2) \quad x = y + 5$$

Si se considera $\phi(y) = \frac{1}{2}(y^2 + 2)$ y $\lambda(y) = y + 5$, la ecuación de la parábola puede escribirse como $x = \phi(y)$ y la ecuación de la recta como $x = \lambda(y)$. Tenga en cuenta el intervalo cerrado $[-2, 4]$ sobre el eje y , y tome una partición de este intervalo. El i -ésimo subintervalo tendrá una longitud de $\Delta_i y$. En el i -ésimo subintervalo $[y_{i-1}, y_i]$ se elige un número w_i . Entonces la longitud del i -ésimo elemento rectangular es de $[\lambda(w_i) - \phi(w_i)]$ unidades y su ancho es de $\Delta_i y$ unidades. La medida del área de la región puede aproximarse mediante la suma de Riemann

$$\sum_{i=1}^n [\lambda(w_i) - \phi(w_i)] \Delta_i y$$

Si A unidades cuadradas es el área de la región, entonces

$$A = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [\lambda(w_i) - \phi(w_i)] \Delta_i y$$

Como λ y ϕ son continuas en $[-2, 4]$, también lo es $\lambda - \phi$, y el límite de la suma de Riemann es una integral definida:

$$\begin{aligned} A &= \int_{-2}^4 [\lambda(y) - \phi(y)] dy \\ &= \int_{-2}^4 [(y + 5) - \frac{1}{2}(y^2 + 2)] dy \\ &= \frac{1}{2} \int_{-2}^4 [-y^2 + 2y + 8] dy \\ &= \frac{1}{2} \left[-\frac{1}{3}y^3 + y^2 + 8y \right]_{-2}^4 \\ &= \frac{1}{2} \left[\left(-\frac{64}{3} + 16 + 32 \right) - \left(\frac{8}{3} + 4 - 16 \right) \right] \\ &= 18 \end{aligned}$$

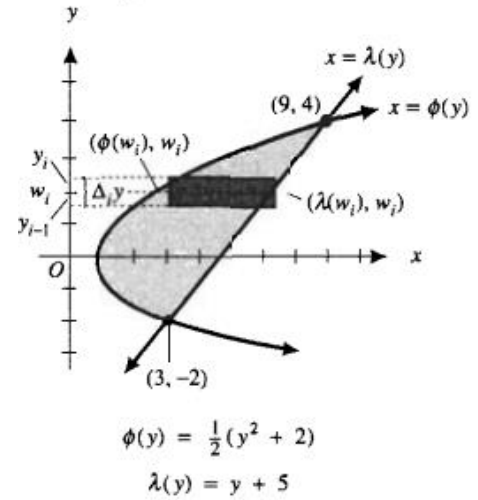


FIGURA 9

► **EJEMPLO 7** Calcule el área de la región limitada por las dos curvas $y = x^3 - 6x^2 + 8x$ y $y = x^2 - 4x$.

Solución Los puntos de intersección de las dos curvas son $(0, 0)$, $(3, -3)$ y $(4, 0)$. En la figura 10 se muestra la región.

Sean

$$f(x) = x^3 - 6x^2 + 8x \quad \text{y} \quad g(x) = x^2 - 4x$$

En el intervalo $[0, 3]$ la curva $y = f(x)$ está por arriba de la curva $y = g(x)$, y en el intervalo $[3, 4]$ la curva $y = g(x)$ se encuentra por arriba de la curva $y = f(x)$. Así, la región debe dividirse en dos regiones separadas R_1 y R_2 , donde R_1 es la región acotada por las dos curvas en el intervalo $[0, 3]$, y R_2 es la región limitada por las dos curvas en el intervalo $[3, 4]$. Si A_1 es el área de R_1 y A_2 es el área de R_2 , entonces

$$A_1 = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [f(w_i) - g(w_i)] \Delta_i x$$

$$A_2 = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [g(w_i) - f(w_i)] \Delta_i x$$

así

$$\begin{aligned} A_1 + A_2 &= \int_0^3 [(x^3 - 6x^2 + 8x) - (x^2 - 4x)] dx \\ &\quad + \int_3^4 [(x^2 - 4x) - (x^3 - 6x^2 + 8x)] dx \\ &= \int_0^3 (x^3 - 7x^2 + 12x) dx + \int_3^4 (-x^3 + 7x^2 - 12x) dx \\ &= \left[\frac{1}{4} x^4 - \frac{7}{3} x^3 + 6x^2 \right]_0^3 + \left[-\frac{1}{4} x^4 + \frac{7}{3} x^3 - 6x^2 \right]_3^4 \\ &= \frac{45}{4} + \frac{7}{12} \\ &= \frac{71}{6} \end{aligned}$$

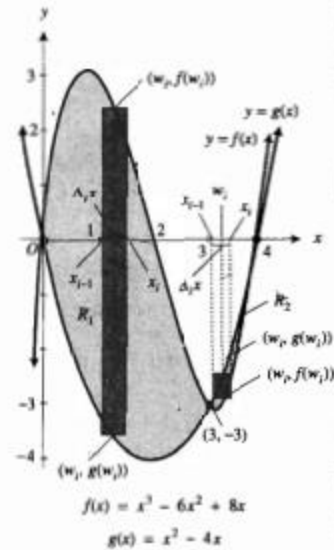


FIGURA 10

Problemas

19. $x^3 = 2y^2; x = 0; y = -2$

21. $y = 2 - x^2; y = -x$

22. $y = x^2; y = x^4$

25. $y = \sqrt{x}; y = x^3$

27. $y^3 = x^2; x - 3y + 4 = 0$

32. $3y = x^3 - 2x^2 - 15x; y = x^3 - 4x^2 - 11x + 30$

35. $y = \cos x - \operatorname{sen} x; x = 0; y = 0$

36. $y = \operatorname{sen} x; y = -\operatorname{sen} x; x = -\frac{1}{2}\pi; x = \frac{1}{2}\pi$

37. $y = |x|; y = x^2 - 1; x = -1; x = 1$

38. $y = |x + 1| + |x|; y = 0; x = -2; x = 3$

40. $y = x^4; y = 4 - x^2$

41. $y = x^2 - 1; y = \operatorname{sen}^2 x$

43. $y = x^3; y = 4 - x^2; \text{el eje } y$

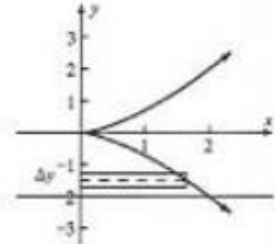
44. $y = x^3; y = 4 - x^2; \text{el eje } x$

Solucionario

19. A square units is the area bounded by $x^3 = 2y^2$; $x = 0$; $y = -2$ in the fourth quadrant. See the figure.

Δ is a partition of the interval $[-2, 0]$ on the y axis.

$$\begin{aligned} A &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \sqrt[3]{2w_i^2} \Delta_i y = \int_{-2}^0 \sqrt[3]{2y^2} dy \\ &= \sqrt[3]{2} \int_{-2}^0 y^{2/3} dy = \sqrt[3]{2} \cdot \frac{3}{5} y^{5/3} \Big|_{-2}^0 \\ &= \frac{3}{5} \sqrt[3]{2} [0 - (-2)^{5/3}] = \frac{3}{5} \sqrt[3]{2} (\sqrt[3]{32}) = \frac{3}{5} \sqrt[3]{64} = \frac{12}{5} \end{aligned}$$



21. A square units is the area of the region bounded by $y = 2 - x^2$; $y = -x$. Set $2 - x^2 = -x$; $0 = x^2 - x - 2 = (x - 2)(x + 1)$. The two curves intersect at the points $(-1, 1)$ and $(2, -2)$. Δ is a partition of the interval $[-1, 2]$ on the x axis.

$$\begin{aligned} A &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [(2 - w_i^2) - (-w_i)] \Delta_i x = \int_{-1}^2 [(2 - x^2) - (-x)] dx = \int_{-1}^2 (-x^2 + x + 2) dx \\ &= -\frac{1}{3} x^3 + \frac{1}{2} x^2 + 2x \Big|_{-1}^2 = \left(-\frac{8}{3} + 2 + 4\right) - \left(-\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{9}{2} \end{aligned}$$

22. A square units is the area bounded by $y = x^2$; $y = x^4$. The region is symmetric. Δ is a partition of the interval $[0, 1]$ on the x axis.

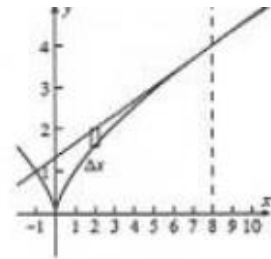
$$A = 2 \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (w_i^2 - w_i^4) \Delta_i x = 2 \int_0^1 (x^2 - x^4) dx = 2 \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_0^1 = 2 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4}{15}$$

25. A square units is the area bounded by $y = \sqrt{x}$ and x^3 meeting at $(0, 0)$ and $(1, 1)$. Δ is a partition of the interval $[0, 1]$ on the x axis.

$$A = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (\sqrt{w_i} - w_i^3) \Delta_i x = \int_0^1 (\sqrt{x} - x^3) dx = \left[\frac{2}{5} x^{3/2} - \frac{1}{4} x^4 \right]_0^1 = \frac{2}{5} - \frac{1}{4} = \frac{5}{12}$$

27. A square units is the area bounded by $y^3 = x^2$; $x = 3y - 4$. When they intersect, $y^3 = (3y - 4)^2$; $0 = y^3 - 9y^2 + 24y - 16 = (y - 1)(y^2 - 8y + 16) = (y - 1)(y - 4)^2$. The two curves cross at $(-1, 1)$ and are tangent at $(8, 4)$. See the figure. Δ is a partition of the interval $[-1, 8]$ on the x axis.

$$\begin{aligned} A &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left[\frac{1}{3}(w_i + 4) - w_i^{2/3} \right] \Delta_i x = \int_{-1}^8 \left[\frac{1}{3}(x + 4) - x^{2/3} \right] dx \\ &= \frac{1}{6} x^2 + \frac{4}{3} x - \frac{3}{5} x^{5/3} \Big|_{-1}^8 = \left(\frac{32}{3} + \frac{32}{3} - \frac{96}{5} \right) - \left(\frac{1}{6} - \frac{4}{3} + \frac{3}{5} \right) = \frac{32}{15} - \left(-\frac{17}{30} \right) = \frac{27}{10} \end{aligned}$$



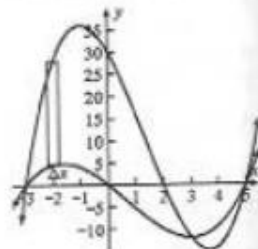
32. $3y = x^3 - 2x^2 - 15x$; $y = x^3 - 4x^2 - 11x + 30$

▷ A sketch of the region R is shown below. Solving the equation $3y = x^3 - 2x^2 - 15x$ for y , we obtain $y = f(x) = \frac{1}{3}x^3 - \frac{2}{3}x^2 - 5x$. Let $g(x) = x^3 - 4x^2 - 11x + 30$. An element of area is a vertical rectangle of width $\Delta_i x$ and height $|f(w_i) - g(w_i)|$. Because

$$\begin{aligned} f(x) - g(x) &= \left(\frac{1}{3}x^3 - \frac{2}{3}x^2 - 5x\right) - (x^3 - 4x^2 - 11x + 30) \\ &= -\frac{2}{3}x^3 + \frac{10}{3}x^2 + 6x - 30 = -\frac{2}{3}(x^3 - 5x^2 - 9x + 45) \\ &= -\frac{2}{3}[x^2(x-5) - 9(x-5)] = -\frac{2}{3}(x+3)(x-3)(x-5) \end{aligned}$$

then $f(x) - g(x) \leq 0$ on $[-3, 3]$ and $f(x) - g(x) \geq 0$ on $[3, 5]$. Thus

$$\begin{aligned} A &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n |f(w_i) - g(w_i)| \Delta_i x \\ &= \int_{-3}^3 -[f(x) - g(x)] dx + \int_3^5 [f(x) - g(x)] dx \\ &= \int_{-3}^3 \left(\frac{2}{3}x^3 - \frac{10}{3}x^2 - 6x + 30\right) dx + \int_3^5 \left[-\frac{2}{3}x^3 + \frac{10}{3}x^2 + 6x - 30\right] dx \\ &= \left[\frac{1}{6}x^4 - \frac{10}{9}x^3 - 3x^2 + 30x\right]_{-3}^3 + \left[-\frac{1}{6}x^4 + \frac{10}{9}x^3 + 3x^2 - 30x\right]_3^5 \\ &= \left[\frac{1}{6}(0) - \frac{10}{9}(27 + 27) - 3(0) + 30(3 + 3)\right] + \left[-\frac{1}{6}(625 - 81) + \frac{10}{9}(125 - 27) + 3(25 - 9) - 30(5 - 3)\right] \\ &= 120 + \frac{56}{9} = 126\frac{2}{9} \end{aligned}$$



• The area of region R is $126\frac{2}{9}$ square units.

35. A square units is the area of the region below $y = \cos x$ and above $y = \sin x$.

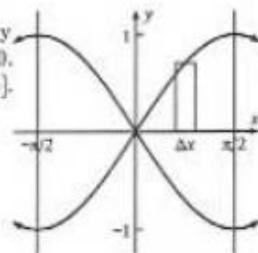
$$\begin{aligned} \Delta &\text{ is a partition of the interval } [0, \frac{1}{4}\pi] \text{ on the } x \text{ axis. } A = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (\cos w_i - \sin w_i) \Delta_i x \\ &= \int_0^{\pi/4} (\cos x - \sin x) dx = \sin x + \cos x \Big|_0^{\pi/4} = \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\right) - (0 + 1) = \sqrt{2} - 1 \end{aligned}$$

36. $y = \sin x$; $y = -\sin x$; $x = -\frac{1}{2}\pi$; $x = \frac{1}{2}\pi$

▷ The figure shows the region R. Let R_1 be the part of R in the first quadrant. By symmetry, the area of R is 4 times the area of R_1 . Let $f(x) = \sin x$ and $g(x) = 0$. Elements of area are vertical rectangles of width $\Delta_i x$ and height $[f(w_i) - g(w_i)]$. Because R_1 is bounded on the left by $x = 0$ and on the right by $x = \frac{1}{2}\pi$, then

$$\begin{aligned} A &= 4 \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [f(w_i) - g(w_i)] \Delta_i x = 4 \int_0^{\pi/2} \sin x dx \\ &= -4 \cos x \Big|_0^{\pi/2} = -4(0 - 1) = 4 \end{aligned}$$

• The area of region R is 4 square units.



37. A square units is the area of the region bounded by $y = |x|$; $y = x^2 - 1$; $x = -1$; $x = 1$.

$$\begin{aligned} A &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [|w_i| - (w_i^2 - 1)] \Delta_i x = \int_{-1}^1 [|x| - (x^2 - 1)] dx = \int_{-1}^0 [-x - (x^2 - 1)] dx + \int_0^1 [x - (x^2 - 1)] dx \\ &= \int_{-1}^0 (-x^2 - x + 1) dx + \int_0^1 (-x^2 + x + 1) dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right]_{-1}^0 + \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_0^1 \\ &= \left[0 - \left(\frac{1}{3} - \frac{1}{2} - 1\right)\right] + \left[\left(-\frac{1}{3} + \frac{1}{2} + 1\right) - 0\right] = \frac{7}{6} + \frac{7}{6} = \frac{7}{3} \end{aligned}$$

38. A square units is the area of the region bounded by $y = |x + 1| + |x|$; $y = 0$; $x = -2$; $x = 3$.

$$\begin{aligned} A &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (|w_i + 1| + |w_i|) \Delta_i x = \int_{-2}^{-1} (-2x - 1) dx + \int_{-1}^0 1 dx + \int_0^3 (2x + 1) dx \\ &= \left[-x^2 - x\right]_{-2}^{-1} + 1 \cdot 1 + \left[x^2 + x\right]_0^3 = 2 + 1 + 12 = 15 \end{aligned}$$

In Exercises 39–48, approximate the area, A sq units, of the region bounded by the graphs of the given equations by doing the following: (a) Plot the graphs in a convenient window and find the x coordinates a and b of the points of intersection to 5 significant digits by using the intersection or trace and zoom-in; (b) express the area of the region as the limit of a Riemann sum; (c) approximate the limit in part (b) to 4 s.d. by using NINT.

40. $y = x^4$; $y = 4 - x^2$

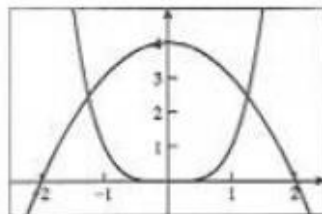
▷ (a) A plot is shown at the right. Using zoom-in, we find $a, b = \pm 1.24962$.

The region is symmetrical.

(b) Δ is a partition of the interval $[0, b]$ on the x axis.

$$A = 2 \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [(4 - w_i^2) - w_i^4] \Delta_i x = 2 \int_0^{1.24962} (4 - x^2 - x^4) dx$$

(c) Using NINT, we find the area to be 7.4772 sq units.



41. $y = x^2 - 1$; $y = \sin^2 x$. $a, b = \pm 1.40449$. The region is symmetrical. Δ is a partition of the interval $[0, b]$ on the

x axis. $A = 2 \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [\sin^2 w_i - (w_i^2 - 1)] \Delta_i x = 2 \int_0^{1.40449} (\sin^2 x - x^2 + 1) dx = 2.20322$

43. $y = x^3$; $y = 4 - x^2$; the y axis. $a = 0$, $b = 1.31460$. Δ is a partition of the interval $[0, b]$ on the x axis.

$$A = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [(4 - w_i^2) - w_i^3] \Delta_i x = \int_0^{1.31460} (4 - x^2 - x^3) dx = 3.7545$$

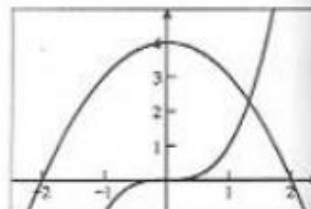
44. $y = x^3$; $y = 4 - x^2$; the x axis

▷ (a) A plot is shown at the right. We see that we need two intervals, $[0, b]$ and $[b, 2]$. Using zoom-in, we find $b = 1.31460$.

(b) Δ is a partition of an interval on the x axis.

$$\begin{aligned} A &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n w_i^3 \Delta_i x + \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (4 - w_i^2) \Delta_i x \\ &= \int_0^{1.31460} x^3 dx + \int_{1.31460}^2 (4 - x^2) dx \end{aligned}$$

(c) Using NINT, we find the area to be $0.74665 + 0.83222 = 1.5789$ sq units.



Teoremas Fundamentales del Cálculo

4.7.1 Primer teorema fundamental del Cálculo

Sea f una función continua en el intervalo cerrado $[a, b]$ y sea x cualquier número de $[a, b]$. Si F es la función definida por

$$F(x) = \int_a^x f(t) dt$$

entonces

$$F'(x) = f(x) \quad (2)$$

$$\Leftrightarrow \frac{d}{dx} \int_a^x f(t) dt = f(x) \quad (3)$$

(Si $x = a$, la derivada en (2) puede ser una derivada por la derecha, y si $x = b$, puede ser una derivada por la izquierda.)

► **EJEMPLO 1** Calcule las derivadas siguientes:

$$(a) \frac{d}{dx} \int_1^x \frac{1}{t^3 + 1} dt \quad (b) \frac{d}{dx} \int_3^{x^2} \sqrt{\cos t} dt$$

Solución

(a) De (3) con $f(t) = \frac{1}{t^3 + 1}$, se tiene

$$\frac{d}{dx} \int_1^x \frac{1}{t^3 + 1} dt = \frac{1}{x^3 + 1}$$

(b) Con $u = x^2$ en la regla de la cadena se obtiene

$$\frac{d}{dx} \int_3^{x^2} \sqrt{\cos t} dt = \frac{d}{du} \int_3^u \sqrt{\cos t} dt \cdot \frac{du}{dx}$$

De (3) con $f(t) = \sqrt{\cos t}$ y como $\frac{du}{dx} = 2x$, se tiene

$$\begin{aligned} \frac{d}{dx} \int_3^{x^2} \sqrt{\cos t} dt &= \sqrt{\cos u} (2x) \\ &= 2x \sqrt{\cos x^2} \end{aligned}$$

4.7.2 Segundo teorema fundamental del Cálculo

Sea f una función continua en el intervalo cerrado $[a, b]$ y sea g una función tal que

$$g'(x) = f(x) \quad (8)$$

para toda x en $[a, b]$. Entonces

$$\int_a^b f(t) dt = g(b) - g(a)$$

(Si $x = a$, la derivada en (8) puede ser una derivada por la derecha, y si $x = b$, la derivada en (8) puede ser una derivada por la izquierda.)

► **EJEMPLO 2** Evalúe

$$\int_{-1}^1 (x^{4/3} + 4x^{1/3}) dx$$

Solución

$$\begin{aligned}\int_{-1}^1 (x^{4/3} + 4x^{1/3}) dx &= \left. \frac{3}{7}x^{7/3} + 4 \cdot \frac{3}{4}x^{4/3} \right|_{-1}^1 \\ &= \frac{3}{7} + 3 - \left(-\frac{3}{7} + 3 \right) \\ &= \frac{6}{7}\end{aligned}$$

► **EJEMPLO 3** Evalúe

$$\int_0^2 2x^2 \sqrt{x^3 + 1} dx$$

Solución

$$\begin{aligned}\int_0^2 2x^2 \sqrt{x^3 + 1} dx &= \frac{2}{3} \int_0^2 \sqrt{x^3 + 1} (3x^2 dx) \\ &= \frac{2}{3} \cdot \left. \frac{(x^3 + 1)^{3/2}}{\frac{3}{2}} \right|_0^2 \\ &= \frac{4}{9} (8 + 1)^{3/2} - \frac{4}{9} (0 + 1)^{3/2} \\ &= \frac{4}{9} (27 - 1) \\ &= \frac{104}{9}\end{aligned}$$

Volúmenes de un sólido en revolución

Método Disco

4.9.2 Teorema

Sea f una función continua en el intervalo cerrado $[a, b]$, y suponga que $f(x) \geq 0$ para toda x en $[a, b]$. Si S es el sólido de revolución obtenido al girar alrededor del eje x la región limitada por la curva $y = f(x)$, el eje x y las rectas $x = a$ y $x = b$, y si V unidades cúbicas es el volumen de S , entonces

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi [f(w_i)]^2 \Delta_i x$$

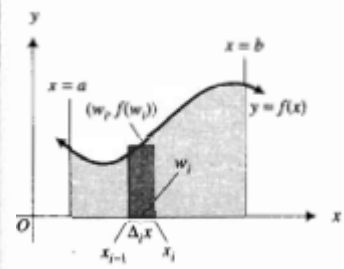
$$= \pi \int_a^b [f(x)]^2 dx$$


FIGURA 12

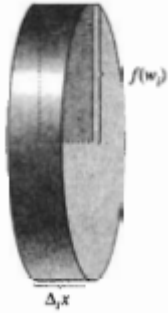


FIGURA 13

EJEMPLO ILUSTRATIVO 2 Calcule el volumen del sólido de revolución generado cuando la región acotada por la curva $y = x^2$, el eje x y las rectas $x = 1$ y $x = 2$ se gira alrededor del eje x . Refiérase a la figura 14, la cual muestra la región y un elemento rectangular de área. La figura 15 presenta el sólido de revolución y un elemento de volumen. La medida del volumen del disco está dado por

Entonces

$$\Delta_i V = \pi (w_i^2)^2 \Delta_i x$$

$$= \pi w_i^4 \Delta_i x$$

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi w_i^4 \Delta_i x$$

$$= \pi \int_1^2 x^4 dx$$

$$= \pi \left(\frac{1}{5} x^5 \right) \Big|_1^2$$

$$= \frac{31}{5} \pi$$

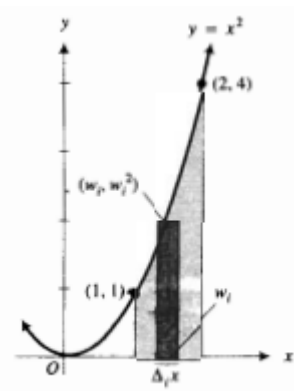


FIGURA 14

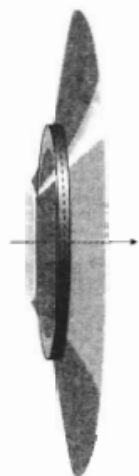


FIGURA 15

Método de Arandelas

4.9.3 Teorema

Sean f y g dos funciones continuas en el intervalo cerrado $[a, b]$ tales que $f(x) \geq g(x) \geq 0$ para toda x en $[a, b]$. Si V unidades cúbicas es el volumen del sólido de revolución generado al girar alrededor del eje x la región limitada por las curvas $y = f(x)$ y $y = g(x)$ y las rectas $x = a$ y $x = b$, entonces

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi([f(w_i)]^2 - [g(w_i)]^2) \Delta_i x$$

$$= \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

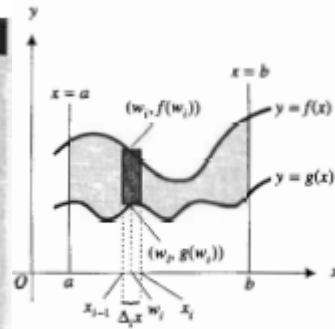


FIGURA 18

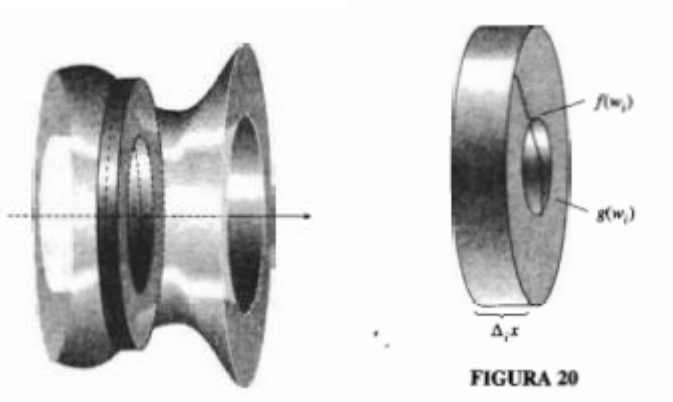
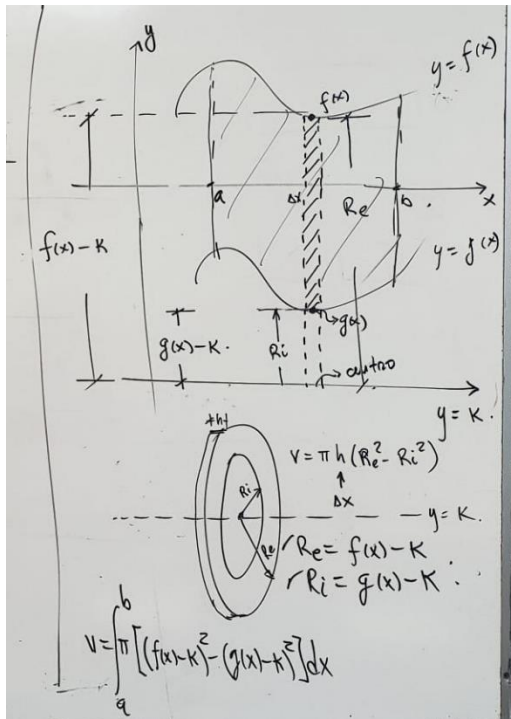


FIGURA 20



► **EJEMPLO 3** Calcule el volumen del sólido generado al girar alrededor del eje x la región acotada por la parábola $y = x^2 + 1$ y la recta $y = x + 3$.

Solución Los puntos de intersección de las dos curvas son $(-1, 2)$ y $(2, 5)$. La figura 21 muestra la región y un elemento rectangular de área. El sólido de revolución y un elemento de volumen se presentan en la figura 22.

Si $f(x) = x + 3$ y $g(x) = x^2 + 1$, entonces la medida del volumen de la arandela circular es

$$\Delta_i V = \pi([f(w_i)]^2 - [g(w_i)]^2) \Delta_i x$$

Si V unidades cúbicas es el volumen del sólido, entonces

$$\begin{aligned} V &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi([f(w_i)]^2 - [g(w_i)]^2) \Delta_i x \\ &= \pi \int_{-1}^2 ((x+3)^2 - (x^2+1)^2) dx \\ &= \pi \int_{-1}^2 [(x+3)^2 - (x^2+1)^2] dx \\ &= \pi \int_{-1}^2 (-x^4 - x^2 + 6x + 8) dx \\ &= \pi \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + 3x^2 + 8x \right]_{-1}^2 \\ &= \pi \left[\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16 \right) - \left(\frac{1}{5} + \frac{1}{3} + 3 - 8 \right) \right] \\ &= \frac{117}{5} \pi \end{aligned}$$

► **EJEMPLO 4** Calcule el volumen del sólido generado al girar alrededor de la recta $x = -4$ la región limitada por las dos parábolas $x = y - y^2$ y $x = y^2 - 3$.

Solución Las curvas se intersectan en los puntos $(-2, -1)$ y $(-\frac{3}{4}, \frac{3}{2})$. La región y un elemento rectangular de área se muestran en la figura 23. La figura 24 presenta el sólido de revolución así como un elemento de volumen, el cual es una arandela.

Sean $F(y) = y - y^2$ y $G(y) = y^2 - 3$. El número de unidades cúbicas del volumen de la arandela circular es

$$\Delta_i V = \pi([4 + F(w_i)]^2 - [4 + G(w_i)]^2) \Delta_i y$$

Por tanto,

$$\begin{aligned} V &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi([4 + F(w_i)]^2 - [4 + G(w_i)]^2) \Delta_i y \\ &= \pi \int_{-1}^{3/2} [(4 + y - y^2)^2 - (4 + y^2 - 3)^2] dy \\ &= \pi \int_{-1}^{3/2} (-2y^3 - 9y^2 + 8y + 15) dy \\ &= \pi \left[-\frac{1}{2}y^4 - 3y^3 + 4y^2 + 15y \right]_{-1}^{3/2} \\ &= \frac{875}{32} \pi \end{aligned}$$

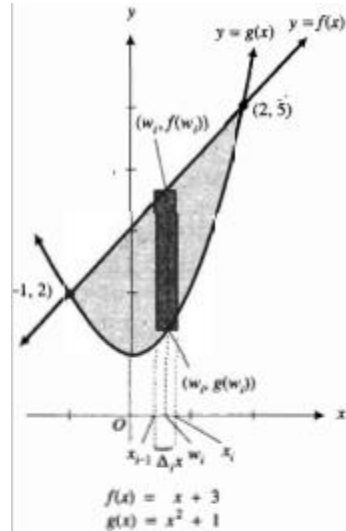


FIGURA 21

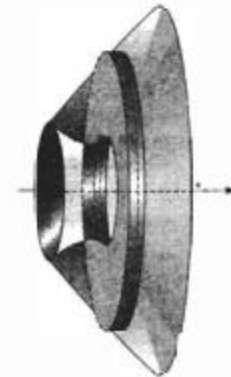


FIGURA 22

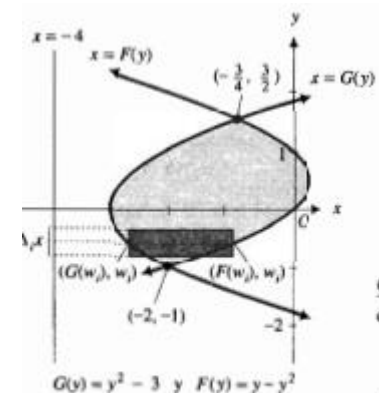


FIGURA 23

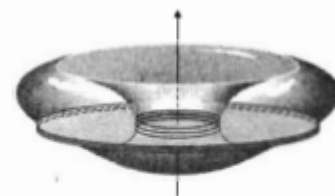


FIGURA 24

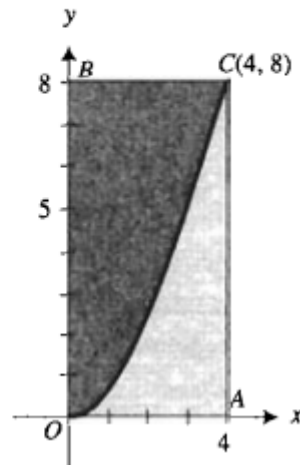
Capas Cilíndricas

$x=h$
 y
 $y=f(x) = x^3 - 2x^2 + 1$
 $y=g(x) = x^{3/2} - 2x + 1$
 $R_m = x - h$
 $h = f(x) - g(x)$
 $V = 2\pi \int_a^b (x-h)[f(x)-g(x)] dx$
 $V = \pi h (R_e^2 - R_i^2)$
 $V = \pi h (R_e + R_i)(R_e - R_i)$
 $t = R_e - R_i = \Delta x$
 $V = 2\pi h \left(\frac{R_e + R_i}{2} \right) t = 2\pi h R_m t$
 $A = \int_a^b [f(x) - g(x)] dx$

Problemas parte 1

En los ejercicios 5 a 12, calcule el volumen del sólido de revolución generado cuando la región de la figura se gira alrededor de la recta indicada. Una ecuación de la curva de la figura es $y^2 = x^3$.

5. OAC alrededor del eje x .
6. OAC alrededor de la recta AC .
7. OAC alrededor de la recta BC .
8. OAC alrededor del eje y .
9. OBC alrededor del eje y .
10. OBC alrededor de la recta BC .
11. OBC alrededor de la recta AC .
12. OBC alrededor del eje x .



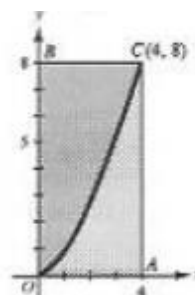
21. La región acotada por la curva $y = \sec x$, el eje x , el eje y y la recta $x = \frac{1}{4}\pi$, se gira alrededor del eje x . Calcule el volumen del sólido generado.
29. Obtenga el volumen del sólido generado al girar alrededor de la recta $x = -4$ la región limitada por esa misma recta y la parábola $x = 4 + 6y - 2y^2$.
35. Determine el volumen del sólido generado cuando la región limitada por el bucle (o lazo) de la curva que tiene la ecuación $x^2y^2 = (x^2 - 9)(1 - x^2)$, se gira alrededor del eje x .
39. Al girar alrededor del eje x la región limitada por la curva $y = \sqrt{2x + 4}$, el eje x , el eje y y la recta $x = c$ ($c > 0$), se generó un sólido de revolución. ¿Para qué valor de c el volumen del sólido será de 12π unidades cúbicas?

Solucionario

In Exercises 5–12, find the volume of the solid of revolution when the given region of the figure is revolved about the indicated axis. An equation of the curve is $y^2 = x^3$.

5. The region is bounded by $y^2 = x^3$, the x axis and $x = 4$. An element of volume is a circular disk centered on the x axis, $x \in [0, 4]$, of radius $w_i^{3/2}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi (w_i^{3/2})^2 \Delta_i x = \pi \int_0^4 x^3 dx = \frac{1}{4} \pi x^4 \Big|_0^4 = 64\pi$$



6. The region is bounded by $y^2 = x^3$, the x axis and $x = 4$. An element of volume is a circular disk centered on $x = 4$, $y \in [0, 8]$, of radius $4 - w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi (4 - w_i^{2/3})^2 \Delta_i y = \pi \int_0^8 (4 - x^{2/3})^2 dx = \pi \int_0^8 (16 - 8x^{2/3} + x^{4/3}) dx = \pi \left[16x - \frac{24}{5} x^{5/3} + \frac{3}{7} x^{7/3} \right]_0^8 \\ = \pi \left(16 \cdot 8 - \frac{24}{5} \cdot 32 + \frac{3}{7} \cdot 128 \right) = \frac{1024}{35} \pi$$

7. The region is bounded by $y^2 = x^3$, the x axis and $x = 4$. An element of volume is a circular ring centered on the line $y = 8$, $x \in [0, 4]$, of radii 8 and $8 - w_i^{3/2}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [\pi(8)^2 - \pi(8 - w_i^{3/2})^2] \Delta_i x = \pi \int_0^4 [64 - (8 - x^{3/2})^2] dx = \pi \int_0^4 (16x^{3/2} - x^3) dx \\ = \pi \left[16 \cdot \frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right]_0^4 = \pi \left[\frac{32}{5} (32) - 64 \right] = \frac{704}{5} \pi$$

8. OAC about the y axis.

> Because the y axis is the axis of revolution, the farther boundary is $x = 4$, and so $f(y) = 4$, and the nearer boundary is the curve $y^2 = x^3$. Solving for x , we have $x = g(y) = y^{2/3}$. An element of volume is a washer centered on the y axis, $y \in [0, 8]$, of radii $f(w_i)$ and $g(w_i)$. By Theorem 4.9.3,

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi [f(w_i)^2 - g(w_i)^2] \Delta_i y = \pi \int_0^8 [4^2 - (y^{2/3})^2] dy \\ = \pi \int_0^8 (16 - y^{4/3}) dy = \pi \left[16y - \frac{3}{7} y^{7/3} \right]_0^8 = \frac{512}{7} \pi$$

- The volume of the solid of revolution is $\frac{512}{7}\pi$ cubic units.

9. The region is bounded by $y^2 = x^3$, the y axis and $y = 8$. An element of volume is a circular disk centered on the y axis, $y \in [0, 8]$, of radius $w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi (w_i^{2/3})^2 \Delta_i y = \pi \int_0^8 y^{4/3} dy = \frac{3}{7} \pi y^{7/3} \Big|_0^8 = \pi \left(\frac{3}{7} \cdot 128 \right) = \frac{384}{7} \pi$$

10. The region is bounded by $y^2 = x^3$, the y axis and $y = 8$. An element of volume is a circular disk centered on $y = 8$, $x \in [0, 4]$, of radius $8 - w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi (8 - w_i^{2/3})^2 \Delta_i x = \pi \int_0^4 (8 - x^{2/3})^2 dx = \pi \int_0^4 (64 - 16x^{2/3} + x^{4/3}) dx = \pi \left[64x - \frac{32}{5} x^{5/3} + \frac{1}{4} x^{7/3} \right]_0^4 \\ = \pi \left(64 \cdot 4 - \frac{32}{5} \cdot 32 + 64 \right) = \frac{576}{5} \pi$$

11. The region is bounded by $y^2 = x^3$, the y axis and $y = 8$. An element of volume is a circular ring centered on the line $x = 4$, $y \in [0, 8]$, of radii 4 and $4 - w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n [\pi(4)^2 - \pi(4 - w_i^{2/3})^2] \Delta_i y = \pi \int_0^8 [16 - (4 - y^{2/3})^2] dy = \pi \int_0^8 (8y^{2/3} - y^{4/3}) dy \\ = \pi \left[8 \cdot \frac{3}{5} y^{5/3} - \frac{3}{7} y^{7/3} \right]_0^8 = \pi \left(\frac{24}{5} \cdot 32 - \frac{3}{7} \cdot 128 \right) = \frac{3456}{35} \pi$$

12. OBC about the x axis.

> Because the x axis is the axis of revolution, the farther boundary is $y = f(x) = 8$ and the nearer boundary is the curve $y^2 = x^3$. Solving for y , we have $y = g(x) = x^{3/2}$. An element of volume is a washer centered on the x axis, $x \in [0, 4]$, of radii $f(w_i)$ and $g(w_i)$ and height $\Delta_i x$. By Theorem 4.9.3,

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi [f(w_i)^2 - g(w_i)^2] \Delta_i x = \pi \int_0^4 [8^2 - (x^{3/2})^2] dx \\ = \pi \int_0^4 (64 - x^3) dx = \pi \left[64x - \frac{1}{4} x^4 \right]_0^4 = \pi [64(4) - \frac{1}{4}(256)] = 192\pi$$

- The volume of the solid of revolution is 192π cubic units.

21. The region is bounded by $y = \sec x$, the x axis, the y axis, and $x = \frac{1}{4}\pi$. An element of volume is a circular disk centered on the x axis, $x \in [0, \frac{1}{4}\pi]$, of radius $\sec w_i$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi(\sec^2 w_i) \Delta_i x = \pi \int_0^{\pi/4} \sec^2 x \, dx = \pi \tan x \Big|_0^{\pi/4} = \pi$$

29. The region is bounded by $x = 4 + 6y - 2y^2$ and $x = -4$. $(4 + 6y - 2y^2) - (-4) = -2(y^2 - 3y - 4) = -2(y+1)(y-4)$ is nonnegative in $[-1, 4]$. An element of volume is a circular disk centered on the line $x = -4$, $y \in [-1, 4]$, of radius $(4 + 6w_i - 2w_i^2) + 4$.

$$\begin{aligned} V &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi(8 + 6w_i - 2w_i^2)^2 \Delta_i y = \pi \int_{-1}^4 (8 + 6y - 2y^2)^2 dy \\ &= \pi \int_{-1}^4 (64 + 96y + 4y^2 - 24y^3 + 4y^4) dy = \pi \left[64y + 48y^2 + \frac{4}{5}y^3 - 6y^4 + \frac{4}{5}y^5 \right] \\ &= \pi \left[(64 \cdot 4 + 48 \cdot 16 + \frac{4}{5} \cdot 64 - 6 \cdot 256 + \frac{4}{5} \cdot 1024) - (-64 + 48 - \frac{4}{5} - 6 - \frac{4}{5}) \right] = \frac{1250}{3}\pi \end{aligned}$$

35. $y^2 = \frac{(x^2 - 9)(1 - x^2)}{x^2}$ is positive in $[-3, -1]$ and $[1, 3]$. Hence the curve consists of two closed loops, each bisected by the x axis; choose the arc in the first quadrant. An element of volume is a circular disk centered

on the x axis, $x \in [1, 3]$, of radius $y(w_i)^2$

$$\begin{aligned} V &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi y(w_i)^2 \Delta_i x = \pi \int_1^3 \frac{(x^2 - 9)(1 - x^2)}{x^2} dx = \pi \int_1^3 (-x^2 + 10 - 9x^{-2}) dx \\ &= \pi \left[-\frac{1}{3}x^3 + 10x + 9x^{-1} \right]_1^3 = \pi \left[(-9 + 30 + 3) - \left(-\frac{1}{3} + 10 + 9\right) \right] = \frac{16}{3}\pi \end{aligned}$$

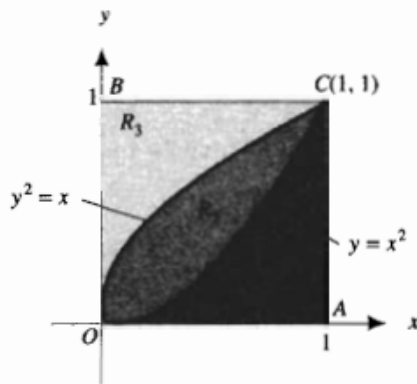
39. The region is bounded by $y = \sqrt{2x + 4}$, the x axis, the y axis, and $x = c$. An element of volume is a circular disk centered on the x axis, $x \in [0, c]$, of radius $\sqrt{2w_i + 4}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi(2w_i + 4) \Delta_i x = \pi \int_0^c (2x + 4) dx = \pi \left[x^2 + 4x \right]_0^c = \pi(c^2 + 4c)$$

Therefore $\pi(c^2 + 4c) = 12\pi$; $c^2 + 4c - 12 = 0$; $(c + 6)(c - 2) = 0$. Because $c > 0$, $c = 2$.

Problemas parte 2

En la figura adjunta, la región limitada por el eje x , la recta $x = 1$ y la curva $y = x^2$ se denota por R_1 ; la región acotada por las curvas $y = x^2$ y $y^2 = x$ se representa mediante R_2 ; y la región limitada por el eje y , la recta $y = 1$ y la curva $y^2 = x$ se denota por R_3 . En los ejercicios 13 a 20, calcule el volumen del sólido generado cuando la región indicada se gira alrededor de la recta dada.



13. R_1 se gira alrededor del eje y ; los elementos rectangulares son paralelos al eje de revolución.
14. Igual que en el ejercicio 13, pero los elementos rectangulares son perpendiculares al eje de revolución.
15. R_2 se gira alrededor del eje x ; los elementos rectangulares son paralelos al eje de revolución.
16. Igual que en el ejercicio 15, pero los elementos rectangulares son perpendiculares al eje de revolución.
17. R_3 se gira alrededor de la recta $y = 2$; los elementos rectangulares son paralelos al eje de revolución.
18. Igual que en el ejercicio 17, pero los elementos rectangulares son perpendiculares al eje de revolución.
19. R_2 se gira alrededor de la recta $x = -2$; los elementos rectangulares son paralelos al eje de revolución.
20. Igual que en el ejercicio 19, pero los elementos rectangulares son perpendiculares al eje de revolución.

En los ejercicios 21 a 24, la región acotada por las curvas $x = y^2 - 2$ y $x = 6 - y^2$ se gira alrededor del eje indicado. Determine el volumen del sólido generado.

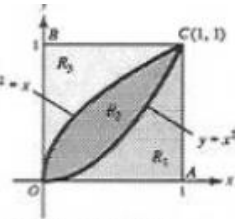
- | | |
|------------------------|------------------------|
| 21. El eje x . | 22. El eje y . |
| 23. La recta $x = 2$. | 24. La recta $y = 2$. |

Solucionario

In Exercises 13–20, find the volume of the solid generated when the indicated region is revolved about the given line.

13. The region is bounded by $y = x^2$, the x axis, and $x = 1$. The rectangular elements are vertical, $x \in [0, 1]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude m_i^2 .

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 2\pi(m_i)m_i^2\Delta_i x = 2\pi \int_0^1 x \cdot x^2 dx = 2\pi \int_0^1 x^3 dx = 2\pi \left[\frac{1}{4}x^4 \right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{1}{2}\pi$$



14. The region is bounded by $y = x^2$, the x axis, and $x = 1$. The rectangular elements are horizontal, $y \in [0, 1]$. An element of volume is a washer centered on the y axis, of radii 1 and $\sqrt{w_i}$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi[1^2 - (\sqrt{w_i})^2]\Delta_i y = \pi \int_0^1 (1 - y) dy = -\frac{1}{2}\pi(1 - y^2) \Big|_0^1 = \frac{1}{2}\pi$$

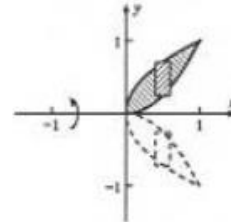
15. The region is bounded by $x = \sqrt{y}$ and $x = y^2$. The rectangular elements are horizontal, $y \in [0, 1]$. An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $\sqrt{m_i} - m_i^2$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 2\pi m_i(\sqrt{m_i} - m_i^2)\Delta_i y = 2\pi \int_0^1 y(y^{1/2} - y^2) dy = 2\pi \int_0^1 (y^{3/2} - y^3) dy = 2\pi \left[\frac{2}{5}y^{5/2} - \frac{1}{4}y^4 \right]_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{4} \right) = \frac{3}{10}\pi$$

16. The region bounded by the curves $y^2 = x$ and $y = x^2$ is revolved about the x axis. The rectangular elements are perpendicular to the axis of revolution.

- The figure shows the region and a plane section of the solid generated. Solving $y^2 = x$ for y , we find the farther boundary is $y = f(x) = \sqrt{x}$ and the nearer boundary is $y = g(x) = x^2$. An element of volume is a washer centered on the x axis, $x \in [0, 1]$, of radii $f(w_i)$ and $g(w_i)$. Thus,

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi[f(w_i)^2 - g(w_i)^2]\Delta_i x = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2] dx = \pi \int_0^1 (x - x^4) dx = \pi \left[\frac{1}{2}x - \frac{1}{5}x^5 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3}{10}\pi$$



- The volume of the solid of revolution is $\frac{3}{10}\pi$ cubic units.

17. The region is bounded by $x = y^2$, the y axis, and $y = 1$. The rectangular elements are horizontal, $y \in [0, 1]$. An element of volume is a cylindrical shell centered on the line $y = 2$, of mean radius $2 - m_i$ and altitude m_i^2 .

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 2\pi(2 - m_i)m_i^2\Delta_i y = 2\pi \int_0^1 (2 - y)y^2 dy = 2\pi \int_0^1 (2y^2 - y^3) dy = 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^1 = 2\pi \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{5}{6}\pi$$

18. The region is bounded by $y = \sqrt{x}$, the y axis, and $y = 1$. The rectangular elements are vertical, $x \in [0, 1]$. An element of volume is a washer centered on the line $y = 2$, of radii $2 - \sqrt{w_i}$ and 1. $V =$

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi[(2 - \sqrt{w_i})^2 - 1^2]\Delta_i x = \pi \int_0^1 [(2 - \sqrt{x})^2 - 1] dx = \pi \int_0^1 (3 - 4x^{1/2} + x) dx = \pi \left[3x - \frac{8}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^1 = \frac{5}{6}\pi$$

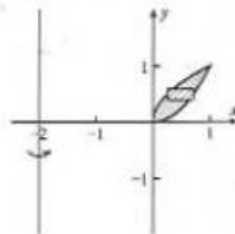
19. The region is bounded by $x = \sqrt{y}$ and $x = y^2$. The rectangular elements are vertical, $x \in [0, 1]$. An element of volume is a cylindrical shell centered on the line $x = -2$, of mean radius $m_i + 2$ and altitude $m_i^{1/2} - m_i^2$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 2\pi(m_i + 2)(m_i^{1/2} - m_i^2)\Delta_i x = 2\pi \int_0^1 (x + 2)(x^{1/2} - x^2) dx = 2\pi \int_0^1 (2x^{1/2} + x^{3/2} - 2x^2 - x^3) dx = 2\pi \left[\frac{4}{3}x^{3/2} + \frac{2}{5}x^{5/2} - \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = 2\pi \left(\frac{4}{3} + \frac{2}{5} - \frac{2}{3} - \frac{1}{4} \right) = \frac{49}{30}\pi$$

20. Find the volume of the solid generated if the region bounded by the curves $y^2 = x$ and $y = x^2$ is revolved about the line $x = -2$. Take rectangular elements that are perpendicular to the axis of revolution.

- The figure shows the region and a plane section of the solid generated. The farther boundary is $y = x^2$ and its distance from $x = -2$ is $f(y) = \sqrt{y} + 2$. The nearer boundary is $y^2 = x$ and its distance from $x = -2$ is $g(y) = x + 2 = y^2 + 2$. An element of volume is a washer centered on $x = -2$, $y \in [0, 1]$, of radii $f(w_i)$ and $g(w_i)$. Therefore,

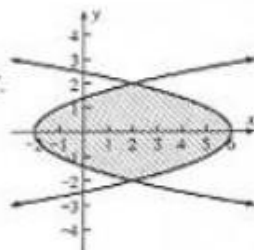
$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi[f(w_i)^2 - g(w_i)^2]\Delta_i y = \pi \int_0^1 [(\sqrt{y} + 2)^2 - (y^2 + 2)^2] dy = \pi \int_0^1 (y + 4\sqrt{y} + 4 - y^4 - 4y^2 - 4) dy = \pi \left[\frac{1}{2}y^2 + \frac{8}{3}y^{3/2} - \frac{1}{5}y^5 - \frac{4}{3}y^3 \right]_0^1 = \pi \left(\frac{1}{2} + \frac{8}{3} - \frac{1}{5} - \frac{4}{3} \right) = \frac{49}{30}\pi$$



- The volume of the solid of revolution is $\frac{49}{30}\pi$ cubic units.

In Exercises 21–24, find the volume of the solid generated by revolving around the indicated line the region bounded by the curves $x = y^2 - 2$ and $x = 6 - y^2$.

- The region is shown at the right. Set $y^2 - 2 = 6 - y^2$; $2y^2 = 8$, $y = \pm 2$



21. the x -axis. The region is symmetric with respect to the x axis; use the upper half. The rectangular elements are horizontal. An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $8 - 2m_i^2$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 2\pi m_i (8 - 2m_i^2) \Delta_i y = 2\pi \int_0^2 y(8 - 2y^2) dy = 2\pi \int_0^2 (8y - 2y^3) dy \\ = 2\pi \left[4y^2 - \frac{1}{2}y^4 \right]_0^2 = 2\pi(16 - 8) = 16\pi$$

22. the y axis. The region is symmetric with respect to the x axis; use the upper half and double. The rectangular elements are horizontal. An element of volume is a washer centered on the y axis, of radii $6 - w_i^2$ and $w_i^2 - 2$.

$$V = \lim_{\|\Delta\| \rightarrow 0} 2 \sum_{i=1}^n \pi [(6 - w_i^2)^2 - (w_i^2 - 2)^2] \Delta_i y = 2\pi \int_0^2 [(36 - 12y^2 + y^4) - (y^4 - 4y^2 + 4)] dy \\ = 2\pi \int_0^2 (32 - 8y^2) dy = 2\pi \left[32y - \frac{8}{3}y^3 \right]_0^2 = 2\pi \cdot \frac{128}{3} = \frac{256}{3}\pi$$

23. the line $x = 2$. The region is symmetric with respect to the line $x = 2$; use the right half.

Method 1. The rectangular elements are vertical, $x \in [2, 6]$. An element of volume is a cylindrical shell of mean radius $m_i - 2$ and altitude $2\sqrt{6 - m_i}$. Let $\sqrt{6 - x} = v$, $x = 6 - v^2$, $dx = -2v dv$. When $x = 2$, $v = 2$; when $x = 6$, $v = 0$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 2\pi(m_i - 2)2\sqrt{6 - m_i} \Delta_i x = 4\pi \int_2^6 (x - 2)\sqrt{6 - x} dx = 4\pi \int_2^0 (4 - v^2)v(-2v dv) \\ = 8\pi \int_0^2 (4v^2 - v^4) dv = 8\pi \left[\frac{4}{3}v^3 - \frac{1}{5}v^5 \right]_0^2 = 8 \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{512}{15}\pi$$

Method 2. The rectangular elements are horizontal, $y \in [-2, 2]$. An element of volume is a circular disk centered on the line $x = 2$ of radius $(6 - w_i^2) - 2$.

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \pi (4 - w_i^2)^2 \Delta_i y = \pi \int_{-2}^2 (4 - y^2)^2 dy = \pi \int_{-2}^2 (16 - 8y^2 + y^4) dy = \pi \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5 \right]_{-2}^2 \\ = \pi \left[\left(32 - \frac{64}{3} + \frac{32}{5} \right) - \left(-32 + \frac{64}{3} - \frac{32}{5} \right) \right] = \frac{512}{15}\pi$$

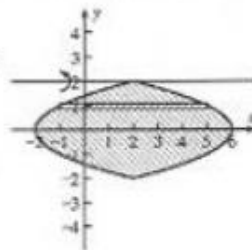
24. the line $y = 2$.

► The figure shows the region R . Because we are given x as a function of y , we take rectangular elements parallel to $y = 2$ at a mean distance of $2 - m_i$. R is bounded on the right by $x = 6 - y^2$ and on the left by $x = y^2 - 2$, a spread of

$$(6 - y^2) - (y^2 - 2) = 8 - 2y^2 = 2(4 - y^2)$$

An element of volume is a cylindrical shell centered on $y = 2$, $y \in [-2, 2]$, of mean radius $2 - m_i$ and height $8 - 2m_i^2$. Therefore,

$$V = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 2\pi(2 - m_i)2(4 - m_i^2) \Delta_i y = 4\pi \int_{-2}^2 (2 - y)(4 - y^2) dy \\ = 4\pi \int_{-2}^2 (8 - 4y - 2y^2 + y^3) dy = 4\pi \left[8y - 2y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4 \right]_{-2}^2 \\ = 4\pi \left[8(2 + 2) - 2(4 - 4) - \frac{2}{3}(8 + 8) + \frac{1}{4}(16 - 16) \right] = 4\pi(32 - \frac{32}{3}) = \frac{256}{3}\pi$$



- The volume of the solid of revolution is $\frac{256}{3}\pi$ cubic units.